



known and widely present in literature, see e.g. [9], while the third appears new, at least to our knowledge.

## 1. A SHORT REVIEW OF MODEL CATEGORIES

For the benefit of the reader and for fixing notation, in this section we briefly recall the notion of model category and the definition of the projective model structure in the category of cochain complexes over a commutative unitary ring. The main reference is Hovey's book [7].

For every category  $\mathbf{C}$  we shall write  $A \in \mathbf{C}$  if  $A$  is an object of  $\mathbf{C}$  and we denote by  $\text{Hom}_{\mathbf{C}}(A, B)$  the set of morphisms  $A \rightarrow B$ . We denote by  $\text{Map}(\mathbf{C})$  the category whose objects are morphism in  $\mathbf{C}$  and whose morphisms are the commutative squares. The following definition gives the basic terminology involved in the notion of model category.

**Definition 1.1.** In the above notation:

- (1) A morphism  $f$  is called a *retract* of a morphism  $g$  if there exists a commutative diagram of the form

$$\begin{array}{ccccc} & & \text{Id}_A & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{\quad} & C & \xrightarrow{\quad} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \xrightarrow{\quad} & D & \xrightarrow{\quad} & B \\ & \searrow & & \nearrow & \\ & & \text{Id}_B & & \end{array} .$$

- (2) A *functorial factorization* is an ordered pair  $(H, K)$  of functors  $\text{Map}(\mathbf{C}) \rightarrow \text{Map}(\mathbf{C})$  such that  $f = K(f)H(f)$  for every  $f \in \text{Map}(\mathbf{C})$ .
- (3) Let  $i \in \text{Hom}_{\mathbf{C}}(A, B)$  and  $p \in \text{Hom}_{\mathbf{C}}(X, Y)$ . We shall say that  $i$  has the *left lifting property (LLP)* with respect to  $p$  and  $p$  has the *right lifting property (RLP)* with respect to  $i$  if for every commutative diagram of solid arrow

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & \nearrow h & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

there exists a morphism  $h: B \rightarrow X$  such that  $hi = f$  and  $ph = g$ .

**Definition 1.2.** A *model structure* on a category  $\mathbf{C}$  is the data of three classes of morphisms called weak equivalences, cofibrations, and fibrations, and two functorial factorizations  $(C, FW)$  and  $(CW, F)$  satisfying the following properties:

- MC1:** (2-out-of-3) If  $f$  and  $g$  are morphisms of  $\mathbf{C}$  such that  $gf$  is defined and two of  $f$ ,  $g$  and  $gf$  are weak equivalences, then so is the third.
- MC2:** (Retracts) If  $f$  and  $g$  are morphisms of  $\mathbf{C}$  such that  $f$  is a retract of  $g$  and  $g$  is a weak equivalence, cofibration, or fibration, then so is  $f$ .
- MC3:** (Lifting) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.
- MC4:** (Factorization) For any morphism  $f$ ,  $C(f)$  is a cofibration,  $FW(f)$  is a trivial fibration,  $CW(f)$  is a trivial cofibration, and  $F(f)$  is a fibration.

A *model category* is a complete and cocomplete category  $\mathbf{C}$  equipped with a model structure.

It is easy to see that in every model category we have, see e.g. [7]:

- every isomorphism is both a trivial fibration and a trivial cofibration;
- the classes of weak equivalences, cofibrations and fibrations are closed by composition;
- the pull-back of a fibration (resp.: trivial fibration) under any morphism is a fibration (resp.: trivial fibration);
- the push-out of a cofibration (resp.: trivial cofibration) under any morphism is a cofibration (resp.: trivial cofibration).

For every commutative unitary ring  $R$  we shall denote by  $\mathbf{coCh}(R)$  the category of cochain complexes of  $R$ -modules. Every object is the data of a collection of  $R$ -modules  $X = \{X^n\}_{n \in \mathbb{Z}}$  and a differential  $d = \{d_n: X^n \rightarrow X^{n+1}\}_{n \in \mathbb{Z}}$ , where each  $d_n$  is an  $R$ -module map and  $d_{n+1}d_n = 0$  for all  $n \in \mathbb{Z}$ . A morphism of cochain complexes  $f: X \rightarrow Y$  is a collection of morphisms of  $R$ -module  $f_n: X^n \rightarrow Y^n$  such that  $d_n f_n = f_{n+1} d_n$ . A quasi-isomorphism of cochain complexes is a morphism that induces isomorphisms on all cohomology groups.

The category  $\mathbf{coCh}(R)$  has all small limits and colimits, which are taken degreewise. The initial and terminal object is the trivial complex, which is 0 in each degree. This category carries several different model structures, [1, 7]: in this paper we only deal with the so called projective model structure on unbounded complexes, although our results can be easily extended also to other model structures and to bounded complexes.

**Theorem 1.3.** *There is a model category structure on the category of chain complexes  $\mathbf{coCh}(R)$  whose*

- *weak equivalences are quasi-isomorphisms;*
- *fibrations are the morphisms that are degreewise epimorphisms;*
- *cofibrations are the morphisms having the left lifting property with respect every trivial fibration;*

*called the projective model structure.*

*Proof.* See e.g. either [7, Thm. 2.3.11] or [1, Thm. 1.4]. □

*Remark 1.4.* There exists a more concrete description of cofibrations as retracts of semifree extensions, see Appendix A. If  $X \rightarrow Y$  is a cofibration then every map  $X^i \rightarrow Y^i$  is injective with projective cokernel; the converse is true if there exists an integer  $n$  such that  $X^i \rightarrow Y^i$  is an isomorphism for every  $i \geq n$ .

## 2. CONTRACTIONS AND ACYCLIC RETRACTIONS

Every cochain complex is intended over a fixed unitary commutative ring  $R$ . If  $N, M$  are cochain complexes and  $n$  is an integer we shall denote by  $\text{Hom}_R^n(N, M)$  the  $R$ -module of sequences  $\{f_i\}_{i \in \mathbb{Z}}$ , where every  $f_i: N^i \rightarrow M^{n+i}$  is a morphism of  $R$ -modules.

**Definition 2.1.** An *acyclic retraction* (AR-data) is a diagram

$$M \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} N$$

where  $M, N$  are cochain complexes of  $R$ -modules and  $\iota, \pi$  are quasi-isomorphisms of cochain complexes such that  $\pi \iota = \text{Id}_M$ . A *morphism* of acyclic retractions

$$f: (M \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\pi} \end{array} N) \rightarrow (A \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} B)$$

is a morphism of cochain complexes  $f: N \rightarrow B$  such that  $f \iota \pi = i p f$ .

Given a morphism of acyclic retractions as in Definition 2.1, we have a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\iota} & N & \xrightarrow{\pi} & M \\ \downarrow \hat{f} & & \downarrow f & & \downarrow \hat{f} \\ A & \xrightarrow{i} & B & \xrightarrow{p} & A \end{array}$$

where  $\hat{f} = pfi$ : in fact  $i\hat{f} = ipfi = f\iota\pi = f\iota$ ,  $\hat{f}\pi = pfi\pi = pipf = pf$ .

**Definition 2.2.** A *contraction* of cochain complexes is a pair  $(M \xrightleftharpoons[\pi]{\iota} N, h)$ , where  $M \xrightleftharpoons[\pi]{\iota} N$  is an acyclic retraction and an element  $h \in \text{Hom}_R^{-1}(N, N)$ , called homotopy, that satisfies the following conditions:

- C1:**  $\iota\pi - \text{Id}_N = d_N h + h d_N$ ;
- C2:**  $\pi h = h \iota = 0$ ;
- C3:**  $h^2 = 0$ .

A morphism of contraction  $f: (M \xrightleftharpoons[\pi]{\iota} N, h) \rightarrow (A \xrightleftharpoons[p]{i} B, k)$  is a morphism of cochain complexes  $f: N \rightarrow B$  such that  $fh = kf$ .

For later use, we point out that if  $(M \xrightleftharpoons[\pi]{\iota} N, h)$  is a contraction, then  $hdh = -h$ , since

$$h + hdh = h + (\iota\pi - \text{Id}_N - dh)h = h - h = 0.$$

Every morphism of contractions is in particular a morphism of acyclic retractions: in the notation of Definition 2.2 we have

$$ipf = (\text{Id}_B + kd + dk)f = f(\text{Id}_N + kd + dk) = f\iota\pi.$$

Thus, denoting by  $\mathbf{AR}(R)$  and  $\mathbf{Contr}(R)$  the categories of acyclic retractions and contractions, we have two forgetful functors

$$\mathbf{Contr}(R) \xrightarrow{\alpha} \mathbf{AR}(R) \xrightarrow{\beta} \mathbf{coCh}(R),$$

$$(M \xrightleftharpoons[\pi]{\iota} N, h) \mapsto M \xrightleftharpoons[\pi]{\iota} N \mapsto N.$$

It is straightforward to check that the categories  $\mathbf{AR}(R)$  and  $\mathbf{Contr}(R)$  are complete and cocomplete. We are now ready to state the main results of this paper.

**Theorem 2.3.** *There exists a model structure on the category  $\mathbf{AR}(R)$  where a morphism  $f$  is a weak equivalence, cofibration, fibration if and only if  $\beta(f)$  is, and the factorizations depends functorially on the factorizations in  $\mathbf{coCh}(R)$ .*

**Theorem 2.4.** *There exists a model structure on the category  $\mathbf{Contr}(R)$  where a morphism  $f$  is a weak equivalence, cofibration, fibration if and only if  $\beta\alpha(f)$  is, and the factorizations depends functorially on the factorizations in  $\mathbf{coCh}(R)$ .*

The proofs will be given in next sections after some preparatory algebraic results about contractions.

### 3. THE BASIC TRICKS

This section is devoted some algebraic properties about contractions and acyclic retractions that will be used in the proof of the main theorems.

**Lemma 3.1** (First basic trick). *Let  $M \xrightleftharpoons[\pi]{\iota} N$  and  $A \xrightleftharpoons[p]{i} B$  be acyclic retractions and let  $f: N \rightarrow B$  be a morphism of cochain complexes. Then*

$$\hat{f} := f - ipf - f\iota\pi + 2ipf\iota\pi: N \rightarrow B$$

*is a morphism of acyclic retractions. Moreover:*

- (1) *the morphism  $f - \hat{f} = ipf(\text{Id} - \iota\pi) + (\text{Id} - ip)f\iota\pi$  induces the trivial morphism in cohomology,*
- (2)  *$\hat{f} = f$  whenever  $f$  is a morphism of acyclic retractions,*
- (3) *if  $g$  is a morphism of acyclic retractions and  $gf$  (resp.:  $fg$ ) is defined, then  $\widehat{gf} = g\hat{f}$  (resp.:  $\widehat{fg} = \hat{f}g$ ).*

*Proof.* Easy and straightforward. □

There exists in literature the the notion of strong deformation data, which lies in an intermediate position with respect to acyclic retractions and contractions.

**Definition 3.2.** A *strong deformation retraction* (SDR-data) of cochain complexes is a pair  $(M \xrightleftharpoons[\pi]{\iota} N, h)$ , where  $M \xrightleftharpoons[\pi]{\iota} N$  is an acyclic retraction and  $h \in \text{Hom}_R^{-1}(N, N)$  satisfy  $\iota\pi - \text{Id}_N = d_N h + h d_N$ . A morphism of strong deformation retractions  $f: (M \xrightleftharpoons[\pi]{\iota} N, h) \rightarrow (A \xrightleftharpoons[p]{i} B, k)$  is a morphism of cochain complexes  $f: N \rightarrow B$  such that  $fh = kf$ .

Thus, if  $\mathbf{SDR}(R)$  denotes the category of strong deformation retractions, we have that  $\mathbf{Contr}(R)$  is a full subcategory of  $\mathbf{SDR}(R)$ . Every morphism of strong deformation retractions is also a morphism a acyclic retractions.

**Lemma 3.3** (Second basic trick). *Let  $(M \xrightleftharpoons[\pi]{\iota} N, h)$  be a strong deformation retraction and denote*

$$D(h) := hd + dh = \iota\pi - \text{Id}_N: N \rightarrow N.$$

*Then the pair*

$$(M \xrightleftharpoons[\pi]{\iota} N, \tilde{h}), \quad \text{where} \quad \tilde{h} = -D(h)hD(h)dD(h)hD(h),$$

*is a contraction. If  $(M \xrightleftharpoons[\pi]{\iota} N, h)$  is already a contraction, then  $h = \tilde{h}$ .*

*Proof.* This is well known [9] and we write the proof only for completeness. We have the equalities

$$dD(h) = D(h)d = dh d, \quad D(h)^2 = (\iota\pi - \text{Id}_N)^2 = \text{Id}_N - \iota\pi = -D(h),$$

$$D(h)\iota = (\iota\pi - \text{Id}_N)\iota = 0, \quad \pi D(h) = \pi(\iota\pi - \text{Id}_N) = 0.$$

Therefore, setting  $k = D(h)hD(h)$ , we get  $k\iota = \pi k = 0$  and

$$dk + kd = dD(h)hD(h) + D(h)hD(h)d = D(h)(dh + hd)D(h) = D(h)^3 = D(h).$$

By definition  $\tilde{h} = -kdk$ , therefore  $\tilde{h}\iota = \pi\tilde{h} = 0$ ,

$$\tilde{h} = -kdk = k(\text{Id}_N - \iota\pi + kd) = k + k^2 d, \quad \tilde{h} = (\text{Id}_N - \iota\pi + dk)k = k + dk^2,$$

and then  $k^2 d = dk^2$ . Finally

$$d\tilde{h} + \tilde{h}d = d(k + dk^2) + (k + k^2 d)d = dk + kd = \iota\pi - \text{Id}_N,$$

$$\tilde{h}^2 = kdkkdk = kd(k^2 d)k = kd(dk^2)k = 0.$$

If  $\pi h = h\iota = h^2 = 0$  then  $D(h)h = hD(h) = -h$  and therefore  $k = D(h)hD(h) = h$ ,  $k^2 = h^2 = 0$ ,  $\tilde{h} = k + dk^2 = k = h$ . □

It is plain that the second basic trick is functorial in the following sense: given a morphism of strong deformation retractions

$$f: (M \xrightleftharpoons[\pi]{\iota} N, h) \rightarrow (A \xrightleftharpoons[p]{i} B, k),$$

then  $\tilde{k}f = f\tilde{h}$ .

**Lemma 3.4** (Third basic trick). *Let  $(M \xrightleftharpoons[\pi]{\iota} N, h)$ ,  $(A \xrightleftharpoons[p]{i} B, k)$  be contractions and let  $f: N \rightarrow B$  be a morphism of acyclic retractions. Then  $\tilde{f} = f - dkfhd$  is a morphism of contractions. Moreover:*

- (1) *the morphism  $f - \tilde{f}$  is homotopic to 0,*
- (2)  *$\tilde{f} = f$  whenever  $f$  is a morphism of contractions,*
- (3) *the transformation  $f \mapsto \tilde{f}$  commutes with compositions.*

*Proof.* We first notice that, since  $dkd = dip - d$  we have

$$dkfhd = dkf(\iota\pi - Id - dh) = dk\iota\pi f - dkf - dkdfh = -dkf - dipfh + dfh = fdh - dkf,$$

and then  $\tilde{f} = f - dkfhd = f + dkf - fdh$ . In particular  $\tilde{f} = f$  whenever  $f$  is already a morphism of contractions.

It is clear that  $d\tilde{f} = \tilde{f}d$ , i.e.,  $\tilde{f}$  is a morphism of complexes, and that the morphism  $f - \tilde{f} = dkfhd = d(kfhd) + (kfhd)d$  is homotopic to 0. Since  $hdh = -h$  and  $kd\pi = -k$  we have

$$k\tilde{f} - \tilde{f}h = kf - fh - kdkfhd + dkfhdh = kf - fh + kfhd - dkfh.$$

Denoting  $\gamma = kf - fh$ , since  $h\iota\pi = 0$  and  $k\iota\pi = 0$  we have  $\gamma\iota\pi = k\iota\pi f = k\iota\pi f = 0$  and

$$\begin{aligned} d\gamma + \gamma d &= dkf + kfd - dfh - fhd = (dk + kd)f - f(dh + hd) \\ &= (ip - I)f - f(\iota\pi - I) = 0. \end{aligned}$$

This implies that  $\tilde{f}$  is a morphism of contractions since

$$\begin{aligned} k\tilde{f} - \tilde{f}h &= \gamma + kfhd - dkfh = \gamma + (\gamma + fh)hd - d(\gamma + fh)h = \gamma + \gamma hd - d\gamma h \\ &= \gamma(I + hd + dh) = \gamma\iota\pi = 0. \end{aligned}$$

If  $(M \xrightleftharpoons[\pi]{\iota} N, h)$ ,  $(A \xrightleftharpoons[p]{i} B, k)$  and  $(P \xrightleftharpoons[q]{j} Q, l)$  are contractions and  $f: N \rightarrow B$ ,  $g: B \rightarrow Q$  are morphisms of acyclic retractions we have

$$\begin{aligned} \tilde{g}\tilde{f} &= (g + dl g - gdk)(f + dkf - fdh) = gf - gdkfhd - dl gkdf \\ &= gf - g(fdh - dkf) - (gdk - dl g)f = gf - gfdh + dl gf = \widetilde{gf}. \end{aligned}$$

□

#### 4. THE PROJECTIVE MODEL STRUCTURE ON ACYCLIC RETRACTIONS

In this section we provide the proof of Theorem 2.3. We first observe that the properties MC1 and MC2 of Definition 1.2 follow immediately from the model structure on  $\mathbf{coCh}(R)$ .

As regards MC3, denote every acyclic retraction  $M \xrightleftharpoons[\pi]{\iota} N$  by the quadruple  $(M, N, \iota, \pi)$ , and consider a commutative diagram of solid arrows in  $\mathbf{AR}(R)$ :

$$\begin{array}{ccc} (M_1, N_1, \iota_1, \pi_1) & \xrightarrow{f} & (M_3, N_3, \iota_3, \pi_3) \\ \downarrow i & \nearrow h & \downarrow p \\ (M_2, N_2, \iota_2, \pi_2) & \xrightarrow{g} & (M_4, N_4, \iota_4, \pi_4) \end{array}$$

where  $i$  is a cofibration (resp.: trivial cofibration) and  $p$  is a trivial fibration (resp.: fibration). The model structure on  $\mathbf{coCh}(R)$  ensures the existence of a morphism of cochain complexes  $h: N_2 \rightarrow N_3$  such that  $hi = f$  and  $ph = g$ . Denoting

$$\hat{h} = h - \iota_3 \pi_3 h - h \iota_2 \pi_2 + 2 \iota_3 \pi_3 h \iota_2 \pi_2: (M_2, N_2, \iota_2, \pi_2) \rightarrow (M_3, N_3, \iota_3, \pi_3),$$

by the first basic trick 3.1, the map  $\hat{h}$  is a morphism in  $\mathbf{AR}(R)$ . Moreover, since  $i, p, f, g$  are morphisms or acyclic retractions, again by Lemma 3.1 we have  $\hat{h}i = \hat{h}i = \hat{f} = f$  and  $p\hat{h} = \widehat{ph} = \hat{g} = g$ , and then  $\hat{h}$  is the required lifting in the category  $\mathbf{AR}(R)$ .

Finally, properties MC4 follows from the following two propositions.

**Proposition 4.1.** *There exists a functorial factorization*

$$(C, FW): \text{Map}(\mathbf{AR}(R)) \rightarrow \text{Map}(\mathbf{AR}(R)) \times \text{Map}(\mathbf{AR}(R))$$

such that  $C(f)$  is a cofibration and  $FW(f)$  is a trivial fibration for every morphism  $f$ .

*Proof.* Consider a morphism  $f$  in  $\mathbf{AR}(R)$  represented by the commutative diagram

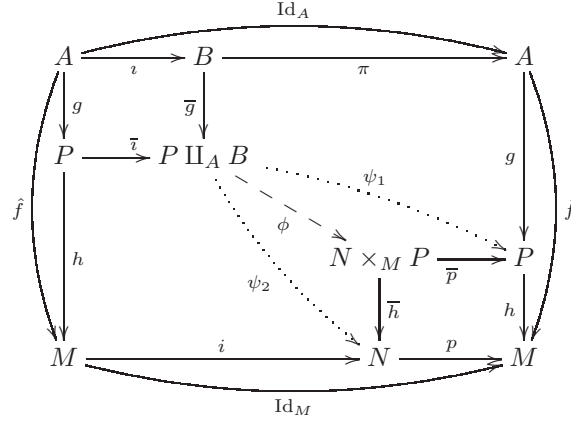
$$\begin{array}{ccccc} & & \text{Id}_A & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & A \\ \downarrow \hat{f} & & \downarrow f & & \downarrow \hat{f} \\ M & \xrightarrow{i} & N & \xrightarrow{p} & M \\ & \searrow & & \nearrow & \\ & & \text{Id}_M & & \end{array}$$

and let  $A \xrightarrow{g} P \xrightarrow{h} M$  be the functorial factorization of  $\hat{f}: A \rightarrow M$  in the model category  $\mathbf{coCh}(R)$ , with  $g$  a cofibration and  $h$  a trivial fibration. Consider now the pushout  $P \amalg_A B$  of  $P \xleftarrow{g} A \xrightarrow{\iota} B$  and the pullback  $N \times_M P$  of  $N \xrightarrow{p} M \xleftarrow{h} P$ . Then we have a commutative diagram

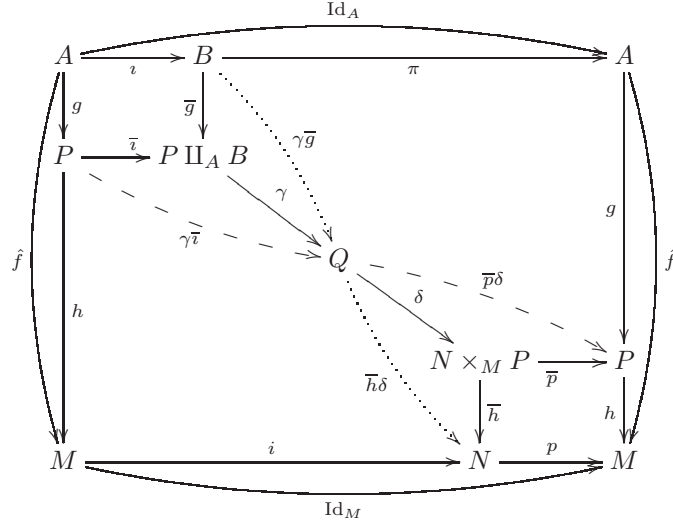
$$\begin{array}{ccccc} & & \text{Id}_A & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & A \\ \downarrow g & & \downarrow \bar{g} & & \downarrow g \\ P & \xrightarrow{\bar{\iota}} & P \amalg_A B & \xrightarrow{f} & P \\ \downarrow \hat{f} & & \downarrow \text{Id}_P & & \downarrow \hat{f} \\ M & \xrightarrow{i} & N & \xrightarrow{p} & M \\ & \searrow & & \nearrow & \\ & & N \times_M P & & \end{array}$$

$\text{Id}_M$

where  $\bar{g}$  is a cofibration and  $\bar{h}$  is a trivial fibration. By the universal property of coproducts, there exists a unique morphism  $\psi_1: P \amalg_A B \rightarrow P$  such that  $g\pi = \psi_1\bar{g}$  and  $\psi_1\bar{i} = \text{Id}_P$ . Similarly, there exists a unique morphism  $\psi_2: P \amalg_A B \rightarrow N$  such that  $\psi_2\bar{g} = f$  and  $\psi_2\bar{i} = ih$ . By the universal property of products, there exists a unique morphism  $\phi: P \amalg_A B \rightarrow N \times_M P$  such that  $\bar{p}\phi = \psi_1$  and  $\bar{h}\phi = \psi_2$ . The above diagram becomes:



Let  $P \amalg_A B \xrightarrow{\gamma} Q \xrightarrow{\delta} N \times_M P$  be the functorial factorization of  $\phi$ , with  $\gamma$  a cofibration and  $\delta$  a trivial fibration. Thus we have a commutative diagram



which reduces to

$$(2) \quad \begin{array}{c} \begin{array}{ccccc} & & \text{Id}_A & & \\ & \curvearrowright & & \curvearrowright & \\ A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & A \\ \downarrow g & & \downarrow \gamma\bar{g} & & \downarrow g \\ P & \xrightarrow{\gamma\bar{i}} & Q & \xrightarrow{\bar{p}\delta} & P \\ \downarrow h & & \downarrow \bar{h}\delta & & \downarrow h \\ M & \xrightarrow{i} & N & \xrightarrow{p} & M \\ & & \text{Id}_M & & \end{array} \\ \end{array} .$$



Since the construction of Diagram (2) is clearly functorial in  $\text{Map}(\mathbf{AR}(R))$ , in order to conclude the proof it is sufficient to prove that the middle row is an acyclic retraction and the middle column is a factorization of  $f$  with  $\gamma\bar{g}$  a cofibration and  $\bar{h}\delta$  a trivial fibration. All of these properties are true because:

- $\bar{p}\delta\gamma\bar{i} = \text{Id}_P$  by construction.
- $\delta$  is a trivial fibration by construction and  $\bar{p}, \bar{h}$  are the pull-backs of the trivial fibrations  $p, h$ . Hence  $\bar{p}\delta, \bar{h}\delta$ , are trivial fibrations and  $\gamma\bar{i}$  is a weak equivalence by the 2 of 3 property.
- $\gamma$  is a cofibration by construction and  $\bar{g}$  is the push-out of the cofibration  $g$ .

□

**Proposition 4.2.** *There exists a functorial factorization*

$$(CW, F): \text{Map}(\mathbf{AR}(R)) \rightarrow \text{Map}(\mathbf{AR}(R)) \times \text{Map}(\mathbf{AR}(R))$$

such that  $CW(f)$  is a trivial cofibration and  $F(f)$  is a fibration for every morphism  $f$ .

*Proof.* Same proof, mutatis mutandis, of Proposition 4.1. □

## 5. THE PROJECTIVE MODEL STRUCTURE ON CONTRACTIONS

In this section we provide the proof of Theorem 2.4: as in the previous section we notice that the properties MC1 and MC2 of Definition 1.2 follow immediately from the model structure on  $\mathbf{coCh}(R)$ .

In order to prove the lifting property MC3 we shall denote every contraction as a pair  $(X, h)$ , where  $X$  is an acyclic retraction and  $h$  is a homotopy related to  $X$  as in Definition 2.2. Consider the following commutative diagram of solid arrow in  $\mathbf{Contr}(R)$ :

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{f} & (C, \gamma) \\ \downarrow i & \nearrow h & \downarrow p \\ (B, \beta) & \xrightarrow{g} & (E, \delta) \end{array}$$

where  $i$  is a cofibration (resp.: trivial cofibration) and  $p$  is a trivial fibration (resp.: fibration). According to Theorem 2.3 there exists a morphism  $h: B \rightarrow C$  of acyclic retractions such that  $hi = f$  and  $ph = g$ . By Lemma 3.4 the morphism  $\tilde{h} = h - d\gamma h\beta d: (B, \beta) \rightarrow (C, \gamma)$  is a morphism of contractions and  $\tilde{h}i = \tilde{h}i = \tilde{f} = f$  and  $p\tilde{h} = p\tilde{h} = \tilde{g} = g$ . This proves property MC3 and the remaining part of this section is devoted to the proof of the factorization property MC4.

**Definition 5.1.** The *path object* functor  $P: \mathbf{coCh}(R) \rightarrow \mathbf{coCh}(R)$  is defined in the following way: for every cochain complex  $B = \{B^i\}$  we have  $P(B)^i = B^i \oplus B^i \oplus B^{i-1}$  and the differential is defined by the formula

$$\delta: P(B)^i \rightarrow P(B)^{i+1}, \quad \delta(a, b, c) = (da, db, a - b - dc).$$

Given  $A, B \in \mathbf{coCh}(R)$ ,  $f, g \in \text{Hom}_R^0(A, B)$  and  $h \in \text{Hom}_R^{-1}(A, B)$ , the linear map

$$A \rightarrow P(B), \quad a \mapsto (f(a), g(a), h(a)),$$

is a morphism of complexes if and only if  $f$  and  $g$  are morphisms of complexes and  $f - g = dh + hd$ .

It follows that the datum  $(M \xrightleftharpoons[\pi]{\iota} N, h)$  is a strong deformation retraction if and only if

$$M \xrightleftharpoons[\pi]{\iota} N \text{ is an acyclic retraction and}$$

$$N \rightarrow P(N), \quad x \mapsto (\iota\pi(x), x, h(x))$$

is a morphism of cochain complexes.

**Lemma 5.2.** *For every cochain complex  $B$ , the natural projection  $P(B) \rightarrow B \oplus B$  is surjective and the inclusion*

$$B \rightarrow P(B), \quad b \mapsto (b, b, 0),$$

*is a quasi-isomorphism. Moreover if  $0 \rightarrow C \rightarrow Q \rightarrow N \rightarrow 0$  is a short exact sequence of cochain complexes, then*

$$0 \rightarrow C[-1] \rightarrow P(Q) \rightarrow (Q \oplus Q) \times_{N \oplus N} P(N) \rightarrow 0$$

*is an exact sequence, where  $C[-1]$  is the cochain complex  $C$  with the degrees shifted by 1.*

*Proof.* Easy and straightforward. □

**Lemma 5.3.** *Let  $f: (A \xrightleftharpoons[p]{i} B, k) \rightarrow (M \xrightleftharpoons[\pi]{i} N, h)$  be a morphism of contractions and let*

$$(A \xrightleftharpoons[p]{i} B) \xrightarrow{\alpha} (P \xrightleftharpoons[q]{j} Q) \xrightarrow{\beta} (M \xrightleftharpoons[\pi]{i} N)$$

*a factorization of  $f$  in the model category  $\mathbf{AR}(R)$  such that  $\alpha$  is a cofibration and  $\beta$  is a fibration. If either  $\alpha$  or  $\beta$  is a weak equivalence, then there exists a homotopy  $l \in \text{Hom}_R^{-1}(Q, Q)$  such that*

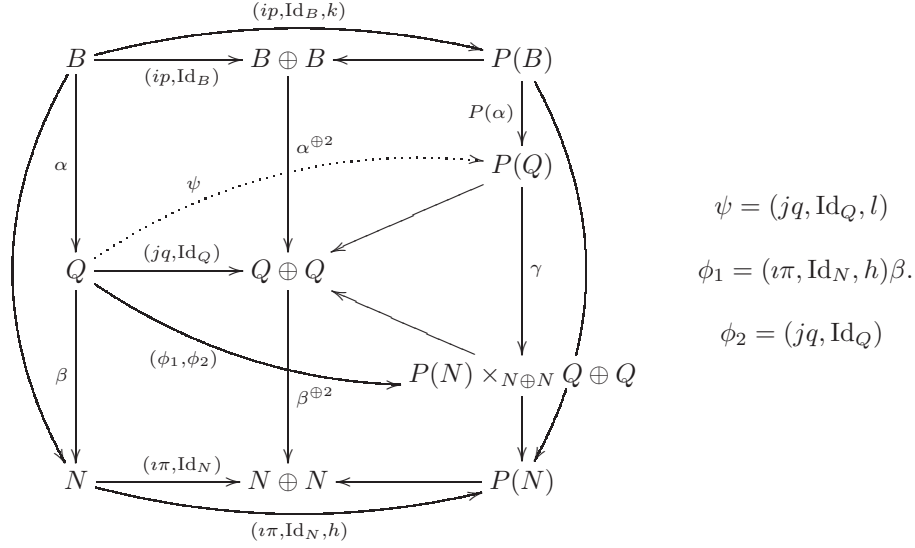
$$(3) \quad (A \xrightleftharpoons[p]{i} B, k) \xrightarrow{\alpha} (P \xrightleftharpoons[q]{j} Q, l) \xrightarrow{\beta} (M \xrightleftharpoons[\pi]{i} N, h)$$

*is a factorization in  $\mathbf{Contr}(R)$ .*

*Proof.* By the second basic trick it is sufficient to prove that there exists  $l$  such that (3) is a factorization of  $f$  in the category of strong deformation retractions. In the model category  $\mathbf{coCh}(R)$  we have a commutative diagram of solid arrows

$$\begin{array}{ccccc}
 & & \xrightarrow{(ip, \text{Id}_B, k)} & & \\
 B & \xrightarrow{(ip, \text{Id}_B)} & B \oplus B & \xleftarrow{\quad} & P(B) \\
 \alpha \downarrow & & \downarrow \alpha^{\oplus 2} & & \downarrow P(\alpha) \\
 Q & \xrightarrow{(jq, \text{Id}_Q)} & Q \oplus Q & \xleftarrow{\quad} & P(Q) \\
 \beta \downarrow & & \downarrow \beta^{\oplus 2} & & \downarrow P(\beta) \\
 N & \xrightarrow{(\imath\pi, \text{Id}_N)} & N \oplus N & \xleftarrow{\quad} & P(N) \\
 & & \xrightarrow{(\imath\pi, \text{Id}_N, h)} & & 
 \end{array}$$

and we want to prove that this diagram can be filled with the dotted arrow. This is equivalent to fill with a dotted arrow the solid commutative diagram



By Lemma 5.2 the morphism  $\gamma$  is a fibration that is trivial if and only if  $\beta$  is a trivial fibration. Therefore the dotted lifting  $\psi$  exists either when  $\alpha$  is a cofibration and  $\beta$  a trivial fibration, or when  $\alpha$  is a trivial cofibration and  $\beta$  a fibration.  $\square$

Finally, properties MC4 follows from the following two propositions.

**Proposition 5.4.** *There exists a functorial factorization*

$$(C, FW): \text{Map}(\mathbf{Contr}(R)) \rightarrow \text{Map}(\mathbf{Contr}(R)) \times \text{Map}(\mathbf{Contr}(R))$$

such that  $C(f)$  is a cofibration and  $FW(f)$  is a trivial fibration for every morphism  $f$ .

*Proof.* For every morphism of contractions  $f: (K, k) \rightarrow (H, h)$ ,  $K, H \in \mathbf{AR}(R)$ , consider the functorial  $(C, FW)$ -factorization  $K \xrightarrow{\alpha} L \xrightarrow{\beta} H$  in the model category  $\mathbf{AR}(R)$  and chose a homotopy  $l$  such that  $(K, k) \xrightarrow{\alpha} (L, l) \xrightarrow{\beta} (H, h)$  is a  $(C, FW)$ -factorization: the existence of  $l$  is provided by Lemma 5.3. This defines two functions  $C, FW$  on the objects of  $\text{Map}(\mathbf{Contr}(R))$ , namely  $C(f) = \alpha$ ,  $FW(f) = \beta$ . Now every morphism  $\phi$  in  $\text{Map}(\mathbf{Contr}(R))$  is given by a commutative square of contractions

$$\begin{array}{ccc} (K_1, k_1) & \xrightarrow{\phi_1} & (K_2, k_2) \\ f_1 \downarrow & & \downarrow f_2 \\ (H_1, h_1) & \xrightarrow{\phi_2} & (H_2, k_2) \end{array}$$

which extends to a commutative diagram of acyclic retractions

$$\begin{array}{ccc} (K_1, k_1) & \xrightarrow{\phi_1} & (K_2, k_2) \\ C(f_1) \downarrow & & \downarrow C(f_2) \\ (L_1, l_1) & \xrightarrow{\psi} & (L_2, l_2) \\ FW(f_1) \downarrow & & \downarrow FW(f_2) \\ (H_1, h_1) & \xrightarrow{\phi_2} & (H_2, k_2) \end{array}$$

and it is sufficient to consider the morphism of contractions  $\tilde{\psi} = \psi - dl_2\psi l_1d$ , provided by Lemma 3.4, in order to have a functorial factorization in the category  $\mathbf{Contr}(R)$ .  $\square$

**Proposition 5.5.** *There exists a functorial factorization*

$$(CW, F): \text{Map}(\mathbf{Contr}(R)) \rightarrow \text{Map}(\mathbf{Contr}(R)) \times \text{Map}(\mathbf{Contr}(R))$$

such that  $CW(f)$  is a trivial cofibration and  $F(f)$  is a fibration for every morphism  $f$ .

*Proof.* Same proof, mutatis mutandis, of Proposition 5.4.  $\square$

## APPENDIX A. SEMIFREE EXTENSIONS

The notion of semifree extension [3, p. 835] extends the classical notion of semifree module and it is very useful in the study of general properties of cofibrations in the projective model structure. This appendix is written for reference purposes and contains results which are well known to experts and in any case easy to prove.

**Definition A.1.** A morphism  $f: C \rightarrow P$  of cochain complexes over a unitary commutative ring  $R$  is called a **semifree extension** if for every  $i \in \mathbb{Z}$  there exists an increasing filtration

$$P_0^i \subset P_1^i \subset P_2^i \subset \dots$$

of  $P^i$  such that:

- (1) every  $P_n^i$  is an  $R$ -submodule of  $P^i$ ,  $\bigcup_{n \geq 0} P_n^i = P^i$  and  $f: C^i \rightarrow P_0^i$  is an isomorphism;
- (2) there exists a direct sum decomposition  $P_{n+1}^i = P_n^i \oplus A_n^i$ , where  $A_n^i$  is a free  $R$ -module, and  $d(A_n^i) \subset P_n^{i+1}$  for every  $n \geq 0$ .

**Example A.2.** Let  $f: C \rightarrow P$  be an injective morphism of cochain complexes such that  $f: C^i \rightarrow P^i$  is an isomorphism for every  $i > 0$  and  $P^i/f(C^i)$  is free for every  $i$ . Then  $f$  is a semifree extension. In fact we can consider the filtration

$$P_n^i = \begin{cases} P^i & \text{if } i + n > 0 \\ f(C^i) & \text{otherwise.} \end{cases}$$

**Theorem A.3.** *Every semifree extension has the left lifting property with respect to every surjective quasi-isomorphism.*

*Proof.* As usual, for every cochain complex  $C$  we shall denote by  $Z(C)$ ,  $B(C)$  and  $H(C)$  the graded modules of cocycles, coboundaries and cohomology of  $C$ .

Let  $C \xrightarrow{f} P$  be a semifree extension,  $X \xrightarrow{g} Y$  a surjective quasi-isomorphism of cochain complexes, and consider a commutative diagram of solid arrows:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & X \\ f \downarrow & \nearrow h & \downarrow g \\ P & \xrightarrow{\beta} & Y \end{array}.$$

Let  $\{P_n\}_{n \in \mathbb{N}}$  be an exhaustive filtration of subcomplexes of  $P$  as in Definition A.1. It is sufficient to define recursively a sequence of liftings

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & X \\ f \downarrow & \nearrow h_n & \downarrow g \\ P_n & \xrightarrow{\beta} & Y \end{array}$$

such that every  $h_n$  extends  $h_{n-1}$  and define  $h$  as the colimit of  $h_n$ . Obviously  $h_0 = \alpha f^{-1}$ ; we may assume  $n \geq 0$  and  $h_n$  already defined.

For every integer  $i$ , there exists a subset  $\{a_j\}_{j \in J_n^i} \subset P_{n+1}^i$  such that  $da_j \in P_n^{i+1}$  and  $P_{n+1}^i$  is the direct summand of  $P_n^i$  and the free module generated by  $\{a_j\}$ . By linearity, in order to define  $h_{n+1}$  which extends  $h_n$  it is sufficient to define the elements  $h_{n+1}(a_j)$  such that  $dh_{n+1}(a_j) = h_n(da_j)$  and  $gh_{n+1}(a_j) = \beta(a_j)$ . Notice that:

- (1)  $d(h_n(da_j)) = h_n(d^2a_j) = 0$ , and therefore  $h_n(da_j) \in Z^{i+1}(X)$ ;
- (2)  $g(h_n(da_j)) = \beta(da_j) = d(\beta(a_j))$ , and therefore  $g(h_n(da_j))$  is trivial in cohomology.

Since  $g$  is a quasi-isomorphism, also  $h_n(da_j)$  is trivial in cohomology and there exists  $x_j \in X^i$  such that  $d(x_j) = h_n(da_j)$ .

Moreover since  $\beta(a_j) - g(x_j) \in Z^i(Y)$  and  $f$  is a surjective quasi-isomorphism there exists  $y_j \in Z^i(X)$  such that  $g(y_j) = \beta(a_j) - g(x_j)$ . It is now sufficient to define  $h_{n+1}(a_j) = x_j + y_j$ .  $\square$

**Theorem A.4.** *Every morphism  $\alpha: C \rightarrow D$  of cochain complexes of  $R$ -modules admits a factorization  $C \xrightarrow{f} P \xrightarrow{g} D$ , with  $f$  is a semifree extension and  $g$  a surjective quasi-isomorphism.*

*Proof.* We construct the factorization by taking an increasing sequence of cochain complexes  $C = P_0 \subset P_1 \subset P_2 \subset \dots$  and a coherent sequence of morphisms of cochain complexes  $g_n: P_n \rightarrow D$ : coherent means that  $g_0 = \alpha$  and every  $g_n$  extends  $g_{n-1}$ . The complexes  $P_n$  and the morphisms  $g_n$  should satisfy the following conditions:

- for every  $i \in \mathbb{Z}$ ,  $P_{n+1}^i = P_n^i \oplus A_n^i$  where  $A_n^i$  is a free  $R$ -module such that  $d(A_n^i) \subset P_n^{i+1}$ . This condition implies that the inclusion  $f: C \rightarrow P = \cup_n P_n$  is a semifree extension.
- $g_1: Z(P_1) \rightarrow Z(D)$  is surjective. This condition implies that  $g = \text{colim } g_n: P \rightarrow D$  is surjective in cohomology.
- $g_2: P_2 \rightarrow D$  is surjective. This condition implies that  $g$  is surjective.
- for every  $n > 2$ ,  $(g_n)^{-1}(B(D)) \cap Z(P_n) \subset B(P_{n+1}) \cap P_n$ . This condition implies that the kernel of  $g_n: H(P_n) \rightarrow H(D)$  is contained in the kernel of  $H(P_n) \rightarrow H(P_{n+1})$  and therefore that  $g$  is injective in cohomology, since  $Z(P) = \cup_n Z(P_n)$ .

The sequence  $(P_n, g_n)$  can be constructed recursively in the following way:

- $n = 0$ : Take  $P_0 = C$  and  $g_0 = \alpha$ .
- $n = 1$ : For every  $i \in \mathbb{Z}$ , let  $A_0^i$  be a free  $R$ -module such that there exists a surjective map  $\pi: A_0^i \rightarrow Z^i(D)$ . Then define  $P_1^i = P_0^i \oplus A_0^i$ ,  $g_1(p + a) = \alpha(p) + \pi(a)$  and  $d(a) = 0$  for every  $a \in A_0^i$ .
- $n = 2$ : For every  $i \in \mathbb{Z}$ , let  $A_1^i$  be a free  $R$ -module such that there exists a surjective map  $\pi: A_1^i \rightarrow D^i$ . If  $\{a_j\}_{j \in J}$  is a basis of  $A_1^i$ , since  $g_1: Z(P_1) \rightarrow Z(D)$  is surjective, there exists a subset  $\{b_j\} \subset P_1^{i+1}$  such that  $g_1(b_j) = d\pi(a_j)$ . Then define  $P_2^i = P_1^i \oplus A_1^i$ ,  $d(a_j) = b_j$  and extend  $g_1$  to the map  $g_2: P_2^i \rightarrow D^i$  by setting  $g_2(a_j) = \pi(a_j)$ .
- $n > 2$ : For every  $i \in \mathbb{Z}$ , let  $A_{n-1}^i$  be a free  $R$ -module such that there exists a surjective morphism  $\delta: A_{n-1}^i \rightarrow g_{n-1}^{-1}(B^{i+1}(D)) \cap Z^{i+1}(P)$ . Then define  $P_n^i = P_{n-1}^i \oplus A_{n-1}^i$ , with differential  $d(x + a) = d(x) + \delta(a)$ , for  $x \in P_{n-1}^i$  and  $a \in A_{n-1}^i$ . If  $\{a_j\}_{j \in J}$  is a basis of  $A_{n-1}^i$ , there exists a subset  $\{c_j\} \subset D^i$  such that  $g_{n-1}\delta(a_j) = dc_j$ . Then we can extend  $g_{n-1}$  to a morphism  $g_n: P_n \rightarrow D$  by setting  $g_n(a_j) = c_j$ .

$\square$

**Corollary A.5.** *In the model structure of cochain complexes where weak equivalences and fibrations are respectively quasi-isomorphisms and surjective maps, a morphism  $g: C \rightarrow D$  is a*

cofibration if and only if there exists a commutative diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & g \swarrow & \downarrow f & \searrow g & \\
 D & \xrightarrow{\quad} & P & \xrightarrow{\quad} & D \\
 & \searrow \text{Id} \swarrow & & & \\
 & & & & 
 \end{array}$$

with  $f$  a semifree extension.

*Proof.* Immediate from the above theorems and the retract argument [7, Lemma 1.1.9].  $\square$

## REFERENCES

- [1] T. Barthel, J.P. May and E. Riehl: *Six model structures for DG-modules over DGAs: model category theory in homological action*. New York J. Math. **20** (2014) 1077-1159; [arXiv:1310.1159](#).
- [2] S. Eilenberg and S. Mac Lane: *On the groups  $H(\pi, n)$ , I*. Ann. of Math. **58** (1953), 55-106.
- [3] Y. Félix, S. Halperin and J. Thomas: *Differential graded algebras in topology*. Handbook of algebraic topology, 829-865, North-Holland, Amsterdam, 1995.
- [4] D. Fiorenza and M. Manetti:  *$L_\infty$  structures on mapping cones*. Algebra Number Theory, **1**, (2007), 301-330; [arXiv:math.QA/0601312](#).
- [5] E. Getzler: *Lie theory for nilpotent  $L_\infty$ -algebras*. Ann. of Math. **170** (1), (2009) 271-301.
- [6] V. K.A.M. Gugenheim: *On the chain-complex of a fibration*. Illinois J. Math. **16** (1972), no. 3, 398-414.
- [7] M. Hovey: *Model categories*. Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, 1999.
- [8] J. Huebschmann and T. Kadeishvili: *Small models for chain algebras*. Math. Z. **207** (1991) 245-280.
- [9] L. Lambe and J. Stasheff: *Applications of perturbation theory to iterated fibrations*. Manuscripta Mathematica **58** (1987), 363-376.
- [10] M. Manetti: *A relative version of the ordinary perturbation lemma*. Rend. Mat. Appl. (7) **30** (2010) 221-238; [arXiv:1002.0683](#).

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